(2) $\mathbb{H}$ is the space of real positive def. quadratic forms.

Let $SP_2 := \{ Y \in M_{2\times 2}(\mathbb{R}) \mid Y > 0, \det Y = 1 \}$

sym. positive def. matrices of det 1.

$g \in SL_2(\mathbb{R})$ acts on $Y \in SP_2$ via

$$G \times SP_2 \longrightarrow SP_2$$
$$(g, Y) \longmapsto g \circ Y := {}^t g^{-1} \, Y g^{-1} = Y[g^{-1}]$$

Any $z \in \mathbb{H}$, define W_z as follows:

$$W_z = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \in SP_2$$

then we have $\qquad g \in SL_2(\mathbb{R})$

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "g"] & \mathbb{H} \\
z \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & gz \arrow[d] \\
W_z \in SP_2 \arrow[r, "g"'] & SP_2 \quad W_{gz} = {}^t g^{-1} W_z g^{-1}
\end{tikzcd}$$

Proof Exercise.

Since we want to study analytic functions on $\mathbb{H}$ which transform in a prescribed way under $SL_2(\mathbb{Z})$ or its s/gps we next look at this group, and the orbits of the group in $\mathbb{H}$.

<u>Defn</u> The group $\Gamma = SL_2(\mathbb{Z})$ is called the <u>modular group</u>

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1 \end{matrix} \right\}$$

Sometimes the image $PSL_2(\mathbb{Z})$ of $SL_2(\mathbb{Z})$ in $PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm I\}$ is also called the (projective) modular group.

There are some special elements of the group: $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$U = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Note in $PSL_2(\mathbb{Z})$ $S^2 = I$, $U^3 = I$

We note the following simple lemma for future use

<u>Lemma 2</u> 1) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ then $\gcd(a,b) = \gcd(c,d) = \gcd(a,c) = \gcd(b,d) = 1$

(ii) If $c, d \in \mathbb{Z}$ s.t $\gcd(c,d) = 1$ then $\exists$ $a, b \in \mathbb{Z}$ s.t $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(iii) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ let $\underline{A} := (c\ d)$. Then $\underline{A} = \underline{B}$ if and only if $\exists$ an $n \in \mathbb{Z}$ s.t $A = T^n B$

Pf (i) clear since $ad - bc = 1$

(ii) If $(c,d) = 1$ then $\exists$ a, b s.t $ad - bc = 1$
Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(iii) ($\Leftarrow$) $\underline{A} = \underline{T^n B} = \underline{T}^n B = \underline{I} B = \underline{B}$

($\Rightarrow$) $\underline{AB^{-1}} = \underline{A} B^{-1} = \underline{B} B^{-1} = (0, 1)$

Hence $\exists$ $n \in \mathbb{Z}$ s.t $AB^{-1} = T^n$

Next we want to determine a set of representatives for the action of Γ on $\mathbb{H}$.

Defn A *fundamental domain* F for a faithful action of a group G on a topological space X is a closed region $F \subset X$ (ie closure of a non-empty open set F°, interior of F) such that

(i) $\bigcup_{g \in G} g(F) = X$ (ii) $F^\circ \cap g(F^\circ) = \phi$ $\forall g \in G \setminus \{e\}$

The family $\{g(F) \mid g \in G\}$ is called a teselation of X.

Added comment: If G does not act faithfully then one can replace G by G/H, where H is the subgroup of elements of G acting trivially on X.

Warning Sometimes Fund. domain is defined as an open set whose closure has the properties (I) and (II)

Fund. domain is a set of representatives for the action of G on X with some extras in the sense that its images cover X without overlaps except possibly at the boundary points.

Example (1) $G = \mathbb{R}$, $X = \mathbb{Z}$

A fund. domain is $[0,1] = \mathbb{R}/\mathbb{Z}$.

(2) Let $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$

$= \{ T^n \mid n \in \mathbb{Z} \}$

acts on $\mathbb{H}$ by translations $z \to z+n$

A fund domain is $\Gamma_\infty \backslash \mathbb{H} = \{ z \in \mathbb{H} \mid 0 \le \mathrm{Re}\, z \le 1 \}$

or $\{ z \in \mathbb{H} \mid -\frac{1}{2} \le \mathrm{Re}\, z \le \frac{1}{2} \}$

Hence fund. domains are not unique

Next we'll write down a fund. domain for $\Gamma\backslash\mathbb{H}$ but first we give a simple Lemma.

Lemma 3: (a) For a point $z \in \mathbb{H}$, there are only finitely many pairs of integers (c, d) such that
$$|cz+d| \leq 1$$

(b) $$\#\left\{(c\ d) \in \mathbb{Z}^2 \;\middle|\; \operatorname{Im}\left(\begin{pmatrix} x & x \\ c & d \end{pmatrix} \circ z\right) > \operatorname{Im} z\right\} < \infty.$$

Proof (a) easy proof:
The set $\{(c,d) \in \mathbb{Z}^2 \mid |cz+d| \leq 1\}$ is compact and discrete hence finite

or Let $(c,d) \in \mathbb{Z}^2$ and $|cz+d| \leq 1$

then $|cz+d|^2 = (cx+d)^2 + c^2y^2 \leq 1$

and $c^2y^2 < (cx+d)^2 + c^2y^2 \leq 1$

Since $z \in \mathbb{H}$, $y > 0$ and hence
$$|c| < \frac{1}{y}$$
Since y is fixed there are only finitely many $c \in \mathbb{Z}$ with $|c| < 1/y$
For any such c, the eqn $(cx+d)^2 + c^2y^2 \leq 1$ also has only finitely many possibilities for

(b) follows from (a) using 1, (15)

$$\operatorname{Im}\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} z\right) = \frac{\operatorname{Im} z}{|cz+d|^2},$$

We now can give a fund domain for $\Gamma \backslash \mathbb{H}$. **Defn** $y = \operatorname{Im} z$ is called the **height** of z.

Thm 4 (a) Every orbit of Γ acting $\mathbb{H}$ contains a point in $F = \{ z \in \mathbb{H} \mid |z| \geq 1, \ |\operatorname{Re} z| \leq \frac{1}{2} \}$

(b) 2 points $z_1, z_2 \in F$ are Γ equivalent $\Leftrightarrow$ either $|\operatorname{Re} z_1| = |\operatorname{Re} z_2| = \frac{1}{2}$ and $z_1 = z_2 \pm 1$
or $|z_1| = |z_2| = 1$ and $z_1 = -1/z_2$

In particular they are points on the boundary of F.

(c) The only pts of F that are fixed by an non-identity element of Γ are

(i) $z = i$ fixed exactly by $\{\pm 1, \pm S\}$

(ii) $z = \rho = e^{2\pi i/3}$ fixed by $\pm\{I, ST, (ST)^2\}$

(iii) $z = -\bar{\rho} = e^{\pi i/3}$ " $\pm\{I, TS, (TS)^2\}$

(v) $z = i\infty$ fixed by the ∞-cyclic gp generated by T.

(d) $\Gamma/_{\pm I}$ is generated by S and T.

Rk

Before we give a proof note that

Γ acts on $\widehat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$.

This is in fact the restriction of its action on the projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

It is natural to adjoin a point at ∞ to $\mathbb{H}$. It is denoted by $i\infty$ since it is in the vertical direction. By definition of the action $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (i\infty) = \frac{a}{c}$ and hence we must also add all rational numbers.

The action of Γ on $\mathbb{Q} \cup \{\infty\}$ is transitive and the orbit of $\{i\infty\}$ is called the cusp of $\Gamma \backslash \widehat{\mathbb{H}}$

If we have a finite index s/gp $\tilde{\Gamma} < \Gamma$. Set of cusps of $\tilde{\Gamma} = \mathbb{Q} \cup \infty / \tilde{\Gamma}$

<u>Proof of thm 4.</u> First of all note that the set $\widehat{F} = F \cup \{i\infty\}$ contains $\{i\infty\}$ whose orbit is $\mathbb{Q} \cup \{i\infty\}$, (if $z \in \mathbb{Q} \cup \{i\infty\}$ write

$z = p/q$ with $(p, q) = 1$ (This includes

$z = i\infty = 1/0$) Since $(p, q) = 1$, $\exists\ a, b \in \mathbb{Z}$ st $ap + bq = 1$. Then $\alpha = \begin{pmatrix} a & b \\ -q & p \end{pmatrix}$ sends z to $i\infty$.

So in fact $\hat{F}$ contains also a representative of the orbit $[i\infty] = \Gamma \backslash \mathbb{Q} \cup \{i\infty\}$.

Now let $z \in \mathbb{H}$, Choose $\gamma_1 \in \Gamma$ so that $\operatorname{Im}(\gamma_1 z)$ is a maximal member of $\{ \operatorname{Im}(\gamma z) \mid \gamma \in \mathbb{Z} \}$ which exists by lemma 3

$\left(\operatorname{Im} \gamma z = \dfrac{\operatorname{Im} z}{|cz+d|^2} \right.$, and $\#\{(c,d) \mid |cz+d| \leq 1\}$ is finite. $\left. \right)$

Now choose n so that the real part of $z^* = T^n(\gamma_1 z) = (\gamma_1 z) + n$ lies in the vertical strip $\{ z \in \mathbb{C} \mid |\operatorname{Re} z| \leq \frac{1}{2} \}$. Hence we have

$|\operatorname{Re} z^*| \leq \frac{1}{2}$.

On the other hand since translation by T^n does not change the imaginary parts

$\operatorname{Im} z^* = \operatorname{Im}(\gamma_1 z)$

Now we cannot have $|z^*| < 1$ since then we would have $\operatorname{Im}\left(-\frac{1}{z^*}\right) = \dfrac{\operatorname{Im} z^*}{|z|^2} > \operatorname{Im} z^*$ contradicting the maximality of height of $\gamma_1 z$.

So z^* satisfies $|z^*| \geq 1$ and $|\operatorname{Re} z^*| \leq \frac{1}{2}$ hence $z^* \in \mathbb{F}$ and $z^* = T^n \gamma_1 z$ is in the orbit of z as claimed in (a)

Rk.

Note in fact that the proof suggests a way to move any given $z \in \mathbb{H}$ into F using only the matrices S, T.
First translate z by T^n so that the real part is in the strip $\{ \tau \mid |\mathrm{Re}\,\tau| \leq \frac{1}{2} \}$. If $T^n z$ is above the unit circle then done. If not use S to get a point with $|\tau| > 1$.
$ST^n z$ now satisfies $|ST^n z| > 1$ but it might be outside the vertical strip. Move it again by a power of T and see if $|T^m S T^n z| > 1$.
If not apply S etc. This process ends again by lemma 3.

(b) Now suppose we have 2 Γ-equivalent
(c) points $z_1, z_2 \in \widehat{F}$ with $\gamma z_1 = z_2$
$\gamma \neq \pm 1$, $\gamma \in \Gamma$. $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
If $z_1 = i\infty$ then $\gamma z_1 \in \mathbb{Q} \cup \{i\infty\}$ for any $\gamma \in \Gamma$. Hence $\gamma z_1 \in \widehat{F}$ has to be $i\infty$. This can only happen (ie $\gamma(i\infty) = i\infty$) if $\gamma = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \pm T^n$ for some $n \in \mathbb{Z}$.

So we can assume $z_1, z_2 \neq i\infty$.

If necessary after replacing γ with γ^{-1} we can assume $\operatorname{Im} z_2 \geq \operatorname{Im} z_1$

Since $\operatorname{Im} z_2 = \operatorname{Im} \gamma z_1 = \dfrac{\operatorname{Im} z_1}{|cz+d|^2}$

this implies $|cz_1+d| \leq 1$.

Now $|cz_1+d| \geq |c| \operatorname{Im} z_1 \geq |c| \frac{\sqrt{3}}{2}$

since $z_1 \in F$.

Hence $|cz_1+d| \leq 1 \Rightarrow 1 \geq |c| \frac{\sqrt{3}}{2}$

Thus $|c| \leq 2/\sqrt{3} < 2$. Hence

$c = 0, \pm 1$.

<u>Case 1</u> $c=0$ then $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = T^b$ $b \in \mathbb{Z}$

$z_2 = z_1 + b$. Since both $z_1, z_2 \in F$

$\operatorname{Re} z_1, \operatorname{Re} z_2 \in [-1/2, 1/2]$ this implies

that either $b=0$, hence $\gamma = I$ or

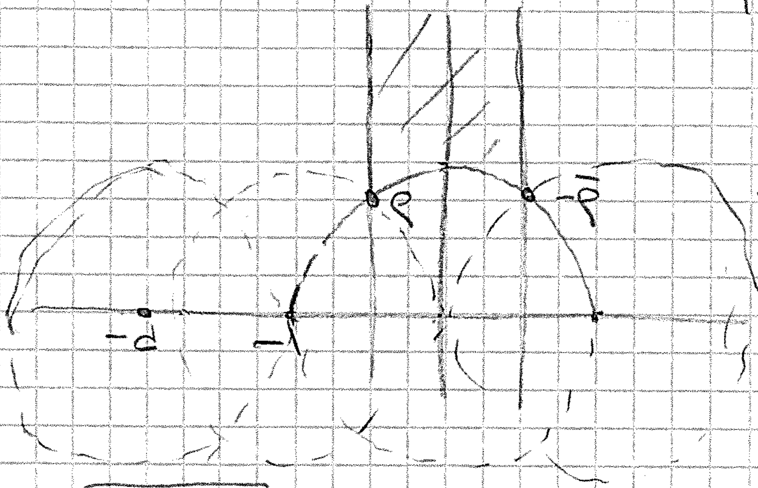
$b = \pm 1$ and $\operatorname{Re} z_1 = \pm \frac{1}{2}$ and $\operatorname{Re} z_2 = \pm \frac{1}{2}$

and $z_2 = z_1 \pm 1$. | ie z_1, z_2 are on the vertical part of F.

<u>Case 2</u> If $c = \pm 1$, since action of γ and $-\gamma$ are the same we can assume wlog $c = 1$.

Then the condition is: $|cz_1+d| = |z_1+d| \leq 1$

This implies z_1 is in the closed disc centered at $-d$ of radius 1.



This disc intersects F only if either $\boxed{d=0}$

or $\boxed{d=1 \text{ and } z_1 = -\bar{\rho}}$

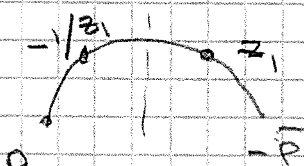
or $\boxed{d=-1 \text{ and } z_1 = \rho.}$

If $\boxed{\begin{matrix} d=0 \\ (c=1) \end{matrix}}$ then we have $\gamma = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$

and $\gamma z_1 = \dfrac{a z_1 - 1}{z_1} = a_1 - \dfrac{1}{z_1} = z_2$

But since $\underset{z_1 \in F}{|z_1| \geq 1}$ and $\underset{|cz_1+d| \leq 1}{|z_1| \leq 1}$ we have

that $|z_1| = 1$ and z_1 is on the circular arc joining ρ to $-\bar{\rho}$. The same is true for $-\frac{1}{z_1}$.



$z_2 = a - \dfrac{1}{z_1} \in F$, $a \in \mathbb{Z}$

implies that

either $\boxed{a=0}$ then $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S$

and $z_2 = -1/z_1$ are the 2

are 2 equivalent points on the arc from ρ to $-\bar{\rho}$.

Note in this case if $z_1 = -1/z_1 = z_2$ then $z = i$

ie $\boxed{\gamma = S \text{ fixes } i}$ but moves the pts

on the right of i to the left of i on the arc.

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$d=0$
$c=1$
If $\boxed{a=1}$ then $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = TS$

Since z_1 is on the circular arc so is $-1/z_1$. Then $z_1 \in F$ and $z_2 = \gamma z_1 = 1 - \frac{1}{z_1} \in F$ forces z_1 to be equal to $-\bar{\rho}$ but then

$$z_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} z_1 = -\bar{\rho} = \rho + 1$$

Hence $\boxed{TS \text{ fixes } -\bar{\rho} = \rho + 1}$.

$d=0, c=1$
If $\boxed{a=-1}$ then $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (ST)^2$

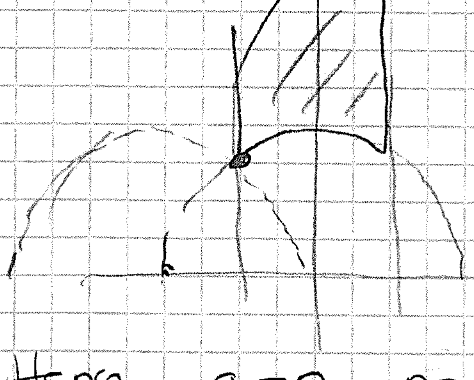
$z_2 = -1 - \frac{1}{z_1} \in F$ forces $z_1 = \rho = z_2$

i.e. $\boxed{(ST)^2 \text{ fixes } \rho}$.

If $\boxed{\begin{matrix} d=1 \\ (c=1) \end{matrix}}$ then $|cz_1 + d| = |z_1 + 1| \leq 1$

Since $|z_1| \in F$ we have $z_1 = \rho$.

and $\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ with $a - b = 1$



$$\gamma z_1 = \gamma \rho = \frac{a\rho + a - 1}{\rho + 1}$$

$$= a - \frac{1}{\rho + 1} = a + \rho = z_2$$

Hence $a = 0$ or $a = 1$. If $a = 0$ then $z_1 = z_2 = \rho$ and $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \boxed{ST \text{ fixes } \rho}$

If $a = 1$ then the two points are ρ and $\rho + 1$

Finally if $\begin{matrix} c=1 \\ d=-1 \end{matrix}$ then $z_1 = -\bar{\rho}$

leads to either $z_1 = -\bar{\rho} = z_2 = \rho$

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = (TS)^2$$

or $z_1 = -\bar{\rho}$, $\gamma = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and $z_2 = \rho$.

Finally to prove (d) let $\Gamma' = \langle S, T \rangle \subset \Gamma$.
Take $z_0 \in \text{int } F$, say $z_0 = 2i$
let $g \in \Gamma$ consider $z = g z_0 \in \mathbb{H}$.

The proof of part (a) shows that $\exists$
$\gamma' \in \Gamma$ such that $\gamma' z \in F$
Then we have $z_0, \gamma' g z_0 \in F$ are equivalent
under Γ. Since $z_0 \in$ int and by part (b),
we must have

$$g\gamma' z_0 = z_0 \Rightarrow g\gamma' = \pm 1$$

$$\Rightarrow g \in \Gamma' \text{ and } \Gamma' = \Gamma.$$

Constructive proof of (d).

One can in fact give an algorithm
which writes any LFT as a word in S and T
and their inverses

It suffices to prove this for matrices
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c \geq 0$.

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We use induction on c.

If $c=0$ then $ad=1$ so $a=d=\pm 1$

and $A = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = T^{\pm b} \in \langle S, T\rangle$

If $c=1$ then $ad-b=1$ so that $b=ad-1$

$$A = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= T^a S T^d$$

Assume the statement holds (ie every $\gamma \in SL_2(\mathbb{Z})/\pm 1$ is a word in S and T) with lower left element $< c$ for some $c \geq 1$.

Since $ad-bc=1$, $(c,d)=1$.

Dividing d by c, $d=cq+r$, $0 < r \lneq c$

Then $A T^{-q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-aq \\ c & r \end{pmatrix}$

$$AT^{-q}S = \begin{pmatrix} -aq+b & -a \\ r & -c \end{pmatrix} \qquad r \lneq c$$

By induction hyp. this last matrix is a word in S and T, hence so is A ∎

(If $c \neq 0$ multiplying by an appropriate power of T and S allows one to reduce the abs. value of c.)

<u>Ex</u>, $\begin{pmatrix} 2 & 9 \\ 3 & 14 \end{pmatrix}$ $\quad 14 = 3.5 - 1$ $\qquad$ 1. 26

$$\begin{pmatrix} 2 & 9 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

$$A\,T^{-5} = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

$$A\,T^{-5}S = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}$$

$$-3 = -1.3 + 0$$

$$A\,T^{-5}S\,T^{-3} = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A\,T^{-5}S\,T^{-3}S = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix} = T^{+1}$$

$$\boxed{A = T^{+1}S\,T^{+3}S\,T^{5}}$$